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# Infinitesimal symmetries and conservation laws of the DNLSE hierarchy and the Noether's theorem

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**Abstract.** The hierarchy of integrable nonlinear equations associated with the quadratic bundle is considered. The expressions for the solution of linearization of these equations and their conservation law in the terms of solutions of corresponding Lax pairs are found. It is shown for the first member of the hierarchy that the conservation law is connected with the solution of linearized equation due to the Noether's theorem. The local hierarchy and three nonlocal ones of the infinitesimal symmetries and conservation laws explicitly expressed through the variables of the nonlinear equations are derived.

**PACS.** 02.30.Jr Partial differential equations – 05.45.Yv Solitons

## 1 Introduction

One of the most effective tools of studying the nonlinear phenomena is the inverse scattering transformation (IST) method [1,2]. This method reduces the solution of the Cauchy initial value problem of nonlinear partial differential equations (PDE's), which admit a representation as the compatibility condition of the overdetermined linear system (Lax pair), to solving linear singular integral equations. It is especially significant that many of PDE's playing the important role in different branches of physics can be investigated in the IST frameworks. For example, the derivative nonlinear Schrödinger equation (DNLSE) that was originally deduced for the Alfvén waves of finite amplitude [3,4] and the equations of massive Thirring model (MTM) [5] belong to the class of the PDE's integrable with the help of IST method for the quadratic bundle [6–8]. It was also revealed that DNLSE describes the behavior of drifting filamentations in nonlinear electrostatic waves of magnetized plasmas [9], light pulses in the optical fibers [10–12], magnetic holes of space plasmas [13] and large-amplitude magnetohydrodynamics waves [14]. The MTM equations were recently shown to appear in the coherent optics and nonlinear acoustics [15] as a limiting case of the system of long/short-wave coupling (see [16] and references therein).

The integrable nonlinear PDE's are well known to possess the infinite hierarchies of infinitesimal symmetries and conservation laws. An existence of them was proposed as the integrability test to characterize the equations solvable by IST (see, e.g., [17,18]). There are different methods of obtaining the infinitesimal symmetries and conservation laws, which originate from the study of KdV equation [19]. Given a set of the scattering data (namely, the time-invariant part of them), one finds the infinite hierarchies of conserved densities [1,2,6,20–22]. The Bäcklund transformation (BT) of integrable equation was used to generate the hierarchy of conservation laws in [23,24]. The approach that exploits the Noether's theorem was applied for the derivation of conservation laws of sine–Gordon and KdV equations [25,26]. To produce the corresponding hierarchy of infinitesimal symmetries, the implicit expressions for the solutions of linearized equations, which are obtained by means of infinitesimal BT, were expanded in the power series on the parameter of this BT. The infinitesimal version of the dressing method was suggested in [27] to construct the infinitesimal symmetries of integrable PDE's. Similar expressions for the perturbations of some nonlinear PDE's and their Lax pairs were presented in [28]. The geometrical approaches that utilize the projective transformations or treat the soliton equations as descriptions of pseudospherical surfaces were developed for nonlinear PDE's associated with matrix Lax pairs of second order in [29,30] (see also [31,32]) and [33], respectively. The hierarchies of local and nonlocal conservation laws of DNLSE were found by using these methods [30,34]. Generalization of these methods on matrix-valued PDE's was suggested in [35]. The theory associated with recursion operators for nonlinear equations in two spatial and one temporal dimensions was developed in [36–38], and, as an application of it, infinitely many symmetries and constants of motion of the Kadomtsev–Petviashvili (KP) and the Davey–Stewartson equations were obtained. The method based on the theory of $\tau$-functions was applied to scalar and two-component KP hierarchies in [39].

Although the methods mentioned above appeal to underlying Lax pair to produce the hierarchies of infinitesimal symmetries and conservation laws, they do not entirely cover the class of PDE's representable as the

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compatibility condition. This concerns especially the cases of reductions of nonlinear PDE's [21] and their integrable deformations (see, e.g., [15,40,41]), which are most interesting from physical point of view. The knowledge of the infinitesimal symmetries and conservation densities of the hierarchy allows one to make sure that the PDE given belongs to it. The approach applicable to all integrable nonlinear equations can be based, for instance, on explicit expressions for the solution of linearized equation and the conservation law in the terms of the solutions of corresponding Lax pairs. In the present report, we construct the infinite hierarchies of local and nonlocal infinitesimal symmetries and conservation laws for the DNLSE hierarchy using such approach.

The paper is organized as follows. The nonlinear equations of the DNLSE hierarchy and their Lax pairs are presented in Section 2 The formulas of expansions in series on the spectral parameter powers of solutions of the Lax pairs are also given there. The solution of linearization of the nonlinear PDE's, which is expressed in the terms of solutions of corresponding Lax pairs, is obtained in Section 3 by means of infinitesimal version of the binary Darboux transformation (DT) [42]. This technique has been applied to the DNLSE [43–45] and for obtaining the infinitesimal symmetries of the nonlinear PDE's [46–48]. To generate the hierarchies of local and nonlocal infinitesimal symmetries expressed explicitly through the variables of the nonlinear equations, the expansions in series of solutions of the Lax pairs or the recursion operator can be used. In Section 4 the conservation law for the DNLSE hierarchy is derived. Substitution of the expansion in series of the Lax pairs solutions into this formula yields the hierarchies of local and nonlocal conservation laws. The connection due to the Noether's theorem of infinitesimal symmetries and conservation laws we found is shown in this section for the DNLSE case.

## 2 The DNLSE hierarchy

Let us consider (direct) Lax pair

$$\psi_x = U(\lambda)\,\psi, \tag{1}$$

$$\psi_t = V(\lambda)\,\psi, \tag{2}$$

where $\psi = \psi(x,t,\lambda) = (\psi_1,\, \psi_2)^T$ is the vector-column solution; $\lambda$ is complex parameter referred to as the spectral parameter in the IST theory; $U(\lambda) = U(x,t,\lambda)$ and $V(\lambda) = V(x,t,\lambda)$ are $2\times 2$ matrix coefficients. The compatibility condition of the overdetermined system (1,2) is

$$U(\lambda)_t - V(\lambda)_x + [\,U(\lambda)\,, V(\lambda)\,] = 0. \tag{3}$$

We suppose in what follows that

$$U(\lambda) = \lambda^2 U^{(2)} + \lambda\, U^{(1)} \tag{4}$$

(i.e., Eq. (1) is the quadratic bundle) and

$$U^{(2)} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^{(1)} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{5}$$

If $V(\lambda)$ is chosen in the form

$$V(\lambda) = \sum_{j=1}^{2m} \lambda^j V^{(j)}, \tag{6}$$

then equation (3) gives us the following expressions for the matrix coefficients of $V(\lambda)$:

$$V^{(2m-2j)} = v^{(2m-2j)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$V^{(2m-2j-1)} = \begin{pmatrix} 0 & v_{12}^{(2m-2j-1)} \\ v_{21}^{(2m-2j-1)} & 0 \end{pmatrix}, \tag{7}$$

$(j = 0, ..., m-1)$, where

$$v^{(2m-2j)} = \partial_x^{-1}(q v_{21}^{(2m-2j-1)} - r v_{12}^{(2m-2j-1)})\,, \tag{8}$$

$$\begin{pmatrix} v_{12}^{(2m-2j-1)} \\ v_{21}^{(2m-2j-1)} \end{pmatrix} = \widehat{R}^{j+1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9}$$

$$\widehat{R} = \frac{1}{2} \begin{pmatrix} i\partial_x + q\partial_x^{-1} r\partial_x & q\partial_x^{-1} q\partial_x \\ r\partial_x^{-1} r\partial_x & -i\partial_x + r\partial_x^{-1} q\partial_x \end{pmatrix}, \tag{10}$$

and system of nonlinear equations

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \partial_x \widehat{R}^m \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11}$$

(Note that operators $\partial_x^{-1}$ in Eqs. (8, 9) for equal $j$'s add the same time-dependent functions as the constants of integration.) To obtain these formulas we make use the identities

$$R\partial_x = \partial_x \widehat{R},$$
$$R^{-1} = 2 \begin{pmatrix} -i + q\partial_x^{-1} r & -q\partial_x^{-1} q \\ -r\partial_x^{-1} r & i + r\partial_x^{-1} q \end{pmatrix} \begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} \tag{12}$$

with operator $R$ being defined as given

$$R = \frac{1}{2} \begin{pmatrix} i\partial_x + \partial_x q\partial_x^{-1} r & \partial_x q\partial_x^{-1} q \\ \partial_x r\partial_x^{-1} r & -i\partial_x + \partial_x r\partial_x^{-1} q \end{pmatrix}. \tag{13}$$

As it will be seen in the next section, $\widehat{R}$ and its adjoint $R$ are the squared eigenfunction operator and the recursion one [17] of the hierarchy considered.

The hierarchy of nonlinear equations (11) was found in [22]. It admits under appropriate choice of the constants of integration the following reduction

$$r = \pm q^*. \tag{14}$$

In this case, the first nontrivial equation of the hierarchy is reduced after rescaling to DNLSE

$$iq_t + q_{xx} \mp i(|q|^2 q)_x = 0. \tag{15}$$

Let us consider the expansions of solutions of equations (1, 2) in the series on the spectral parameter powers. In the neighborhood of point $\lambda = \infty$, vector solutions of the Lax pairs of nonlinear equations (11) are represented as

$$\psi = \sum_{k=0}^{\infty} \frac{A^{(k)}}{\lambda^k} M\,|a\rangle. \qquad (16)$$

Here $|a\rangle$ is a constant vector-column,

$$M = \begin{pmatrix} \mathrm{e}^{-i\lambda^2 x + \lambda^{2m} v^{(2m)} t} & 0 \\ 0 & \mathrm{e}^{i\lambda^2 x - \lambda^{2m} v^{(2m)} t} \end{pmatrix}$$

and coefficients $A^{(k)}$ solve system of equations

$$\begin{cases} [A^{(k)}, U^{(2)}] + A_x^{(k-2)} = U^{(1)} A^{(k-1)} \\ [A^{(k)}, V^{(2m)}] + A_t^{(k-2m)} = \sum\limits_{j=1}^{2m-1} V^{(2m-j)} A^{(k-j)} \end{cases}.$$

An expansion in series of the solutions of Lax pairs considered in the neighborhood of point $\lambda = 0$ has form

$$\psi = \sum_{k=0}^{\infty} \lambda^k B^{(k)}\,|a\rangle, \qquad (17)$$

where $B^{(0)} = E$ and coefficients $B^{(k)}$ $(k \geq 1)$ are determined from equations

$$\begin{cases} B_x^{(k)} = U^{(2)} B^{(k-2)} + U^{(1)} B^{(k-1)} \\ B_t^{(k)} = \sum\limits_{j=1}^{2m} V^{(j)} B^{(k-j)} \end{cases}.$$

It is seen that the first coefficients of the expansions are

$$A^{(0)} = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \quad A^{(1)} = \frac{i}{2}\begin{pmatrix} 0 & -qw^{-1} \\ rw & 0 \end{pmatrix},$$

$$A^{(2)} = \frac{1}{8}\begin{pmatrix} w\int^x (2qr_x + iq^2r^2)\,dx & 0 \\ 0 & w^{-1}\int^x (2q_x r - iq^2r^2)\,dx \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \; B^{(2)} = \begin{pmatrix} -ix + \int^x qv\,dx & 0 \\ 0 & ix + \int^x ru\,dx \end{pmatrix},$$

where

$$u = \int^x q\,dx, \quad v = \int^x r\,dx, \quad w = \exp\Big(i\int^x qr/2\,dx\Big).$$

## 3 Darboux transformation and infinitesimal symmetries

The hierarchy of nonlinear equations (11) follows also from the compatibility condition of dual Lax pair

$$\xi_x = -\xi\,U(æ), \qquad (18)$$

$$\xi_t = -\xi\,V(æ), \qquad (19)$$

where $\xi = \xi(x,t,æ) = (\xi_1,\,\xi_2)$ is a vector-row solution, æ is the spectral parameter of the dual pair. Since matrix coefficients $U(\lambda)$ and $V(\lambda)$ defined by equations (4–7) satisfy conditions

$$\sigma_1 U(-\lambda) + U(\lambda)^T \sigma_1 = 0, \;\; \sigma_1 V(-\lambda) + V(\lambda)^T \sigma_1 = 0, \; (20)$$

where $\sigma_1$ is the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the connection between the solutions of systems (1, 2) and (18, 19) exists:

$$\xi = \psi^T \sigma_1, \quad æ = -\lambda. \qquad (21)$$

In the case of reduction (14) the solutions with complex conjugate spectral parameters are also connected. For instance, $(\psi_2^*, \pm\psi_1^*)^T$ is a solution of direct Lax pair (1, 2) with spectral parameter $\lambda^*$.

Let vector-column $\varphi = (\varphi_1, \varphi_2)^T$ and vector-row $\chi = (\chi_1, \chi_2)$ be the solutions of Lax pairs (1, 2) and (18, 19) with spectral parameters $\mu$ and $\nu$, respectively. The Lax pairs are covariant with respect to "turned" binary Darboux transformation (BDT) $\{\psi,\, \xi,\, U(\lambda),\, V(\lambda)\} \to \{\psi[1],\, \xi[1],\, U(\lambda)[1],\, V(\lambda)[1]\}$ of the form

$$\psi[1] = gT(\lambda)\psi, \quad \xi[1] = \xi T(æ)^{-1} g^{-1}, \qquad (22)$$

$$U(\lambda)[1] = \lambda^2 U^{(2)}[1] + \lambda\, U^{(1)}[1], \;\; V(\lambda)[1] = \sum_{j=1}^{2m} \lambda^j V^{(j)}[1], \qquad (23)$$

where

$$T(\lambda) = E - \frac{\mu-\nu}{\lambda-\nu} P = \Big(1 - \frac{\nu-\mu}{\lambda-\mu} P\Big)^{-1},$$

$$P = \frac{\varphi\chi}{\chi\varphi}, \quad g = \sigma_1 T(0)^{-1}$$

and

$$U^{(2)}[1] = gU^{(2)}g^{-1}, \quad V^{(2m)}[1] = gV^{(2m)}g^{-1}, \qquad (24)$$

$$U^{(1)}[1] = g\Big(U^{(1)} + (\mu-\nu)[U^{(2)}, P]\Big)g^{-1}, \qquad (25)$$

$$V^{(j)}[1] = gV^{(j)}g^{-1} + (\mu-\nu) \times \sum_{k=j+1}^{2m} \nu^{k-j-1}\Big(V^{(k)}[1]gP - gP\,V^{(k)}\Big)g^{-1} \qquad (26)$$

$(j = 1, ..., 2m-1)$. We call this transformation as "turned" because formulas (22–26) is a product of usual BDT [46,47] and additional gauge transformation carried out with the help of matrix $g$. This additional transformation allows us to avoid an appearance of the terms at the zero power of $\lambda$ in the expressions for $U(\lambda)[1]$ and $V(\lambda)[1]$ (compare (23) with (4) and (6)).

Conditions (20, 21) are fulfilled for transformed matrix coefficients $U(\lambda)[1]$, $V(\lambda)[1]$ and solutions $\psi[1]$, $\xi[1]$ of the transformed Lax pairs if we impose restriction

$$\chi = \varphi^T \sigma_1, \quad \nu = -\mu.$$

In this case, we have

$$U^{(2)}[1] = U^{(2)},$$

$$V^{(2m)}[1] = V^{(2m)}.$$

Then, equation (25) gives us expressions for new (transformed) solutions of hierarchy of nonlinear equations (11):

$$q[1] = r - \frac{1}{\mu}\left(\frac{\varphi_2}{\varphi_1}\right)_x,$$

$$r[1] = q - \frac{1}{\mu}\left(\frac{\varphi_1}{\varphi_2}\right)_x.$$

The second iteration of the BDT (22–26) keeping conditions (20, 21) yields the following formulas

$$q[2] = q - \frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2}\left(\frac{\varphi_1^{(1)}\varphi_1^{(2)}}{\mu_1\varphi_1^{(1)}\varphi_2^{(2)} - \mu_2\varphi_2^{(1)}\varphi_1^{(2)}}\right)_x, \quad (27)$$

$$r[2] = r + \frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2}\left(\frac{\varphi_2^{(1)}\varphi_2^{(2)}}{\mu_2\varphi_1^{(1)}\varphi_2^{(2)} - \mu_1\varphi_2^{(1)}\varphi_1^{(2)}}\right)_x, \quad (28)$$

where $\varphi_1^{(k)}$ and $\varphi_2^{(k)}$ are the components of vector solution $\varphi^{(k)}$ of the direct Lax pair with spectral parameters $\mu_k$ ($k = 1, 2$). If we put here $\varphi^{(2)} = ({\varphi_2^{(1)}}^*, \pm{\varphi_1^{(1)}}^*)^T$ and $\mu_2 = \mu_1^*$, then

$$r[2] = \pm q[2]^*.$$

This way we come to DT for the DNLSE hierarchy. The compact form of $N$th iteration of this transformation is presented in [45].

Considering limits $\mu_1 \to \mu$ and $\mu_2 \to \mu$ in equations (27, 28), one obtains the following expressions (up to a multiplier) for solution of the linearization of system (11):

$$\delta q = \left(\varphi_1^{(1)}\varphi_1^{(2)}\right)_x, \quad (29)$$

$$\delta r = -\left(\varphi_2^{(1)}\varphi_2^{(2)}\right)_x. \quad (30)$$

It is checked by straightforward calculation that

$$R\begin{pmatrix}\delta q\\ \delta r\end{pmatrix} = \mu^2\begin{pmatrix}\delta q\\ \delta r\end{pmatrix}. \quad (31)$$

This identity allows us to define in a recurrent manner the coefficients of expansions of the right–hand sides of equations (29, 30) in the power series on the spectral parameter at a neighborhood of the points $\mu = \infty$ and $\mu = 0$. The coefficients of these expansions

$$\delta q = \sum_{k=0}^{\infty}\frac{\delta q^{(k)}}{\mu^{2k}}, \quad \delta r = \sum_{k=0}^{\infty}\frac{\delta r^{(k)}}{\mu^{2k}}$$

and

$$\delta q_j = \sum_{k=0}^{\infty}\mu^{2k}\delta q_j^{(k)}, \quad \delta r_j = \sum_{k=0}^{\infty}\mu^{2k}\delta r_j^{(k)}$$

form the infinite hierarchies of infinitesimal symmetries. Operator $R$ satisfying (31) is nothing but the recursion operator of the hierarchy (11). In the case of point $\mu = 0$, there exist three hierarchies of nonlocal infinitesimal symmetries $\delta q_j^{(k)}$, $\delta r_j^{(k)}$ ($j = 1, 2, 3,\ k = 0, 1, ...$) that correspond to different choices of the constants of integration in operator $R^{-1}$ (see Eq. (12)). The first nontrivial members of the hierarchies for the points $\mu = \infty$ and $\mu = 0$, respectively, are

$$\begin{cases}\delta q^{(1)} = q_x\\ \delta r^{(1)} = r_x\end{cases}, \quad \begin{cases}\delta q^{(2)} = q_t/2\\ \delta r^{(2)} = r_t/2\end{cases},$$

$$\begin{cases}\delta q^{(3)} = (-q_{xx} + 3iq_x qr + 3q^3r^2/2)_x/4\\ \delta r^{(3)} = (-r_{xx} - 3ir_x qr + 3q^2r^3/2)_x/4\end{cases}$$

and

$$\begin{cases}\delta q_1^{(1)} = q\\ \delta r_1^{(1)} = -r\end{cases}, \quad \begin{cases}\delta q_2^{(1)} = 2(qv - i)\\ \delta r_2^{(1)} = -2rv\end{cases},$$

$$\begin{cases}\delta q_3^{(1)} = 2qu\\ \delta r_3^{(1)} = -2(ru + i)\end{cases}.$$

It is seen from these formulas that $\delta q^{(k)} \sim v_{12,x}^{(2m-2k+1)}$, $\delta r^{(k)} \sim v_{21,x}^{(2m-2k+1)}$ and the infinite hierarchy corresponding to the point $\mu = \infty$ is local. Another way of producing the hierarchies of infinitesimal symmetries is to substitute expansions (16) and (17) into equations (29, 30).

## 4 Conservation laws and Noether's theorem

Let us consider identity

$$(\xi\psi)_{xt} = (\xi\psi)_{tx}.$$

Excluding here the derivatives of $\psi$ and $\xi$ on $x$ in the left-hand side and the derivatives on $t$ in the right-hand side with the help of equations (1, 2) and (18, 19), respectively, and dividing the relation obtained on $\lambda - æ$, we come to the conservation law of the DNLSE hierarchy

$$T_t + X_x = 0,$$

where

$$T = \xi\Big((\lambda + æ)U^{(2)} + U^{(1)}\Big)\psi, \quad (32)$$

$$X = -\xi\sum_{k=1}^{2m}\sum_{j=0}^{k-1}\lambda^{k-j-1}æ^j V^{(k)}\psi. \quad (33)$$

If we put $\lambda = æ = \mu$, $\psi = \varphi^{(1)}$ and $\xi = (\varphi_2^{(2)}, -\varphi_1^{(2)})$, where vectors $\varphi^{(k)} = (\varphi_1^{(k)}, \varphi_2^{(k)})^T$ $(k = 1, 2)$, as it was supposed at the end of the previous section, are solutions of Lax pair (1, 2) with spectral parameter $\mu$, then expressions (32, 33) are rewritten in the following manner

$$T = -2i\mu(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)}) + q\varphi_2^{(1)}\varphi_2^{(2)} - r\varphi_1^{(1)}\varphi_1^{(2)}, \tag{34}$$

$$X = -2\sum_{k=1}^{m} k\mu^{2k-1}v^{(2k)}(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)}) + \sum_{k=1}^{m}(2k-1)\mu^{2k-2}\Big(v_{21}^{(2k-1)}\varphi_1^{(1)}\varphi_1^{(2)} - v_{12}^{(2k-1)}\varphi_2^{(1)}\varphi_2^{(2)}\Big). \tag{35}$$

Substitution of expansions (16) and (17) of the Lax pair solutions at the neighborhood of points $\mu = \infty$ and $\mu = 0$ into these formulas leads to the hierarchies of the conservation laws expressed explicitly through the solutions of nonlinear equations (11). For example, in the case of point $\mu = 0$ we have three infinite hierarchies $T_{j,t}^{(k)} + X_{j,x}^{(k)} = 0$ $(j = 1, 2, 3,\ k = 0, 1, ...)$, whose first conserved densities and currents are

$$T_1^{(0)} = q,\ X_1^{(0)} = -v_{12}^{(1)}, \quad T_2^{(0)} = r,\ X_2^{(0)} = -v_{21}^{(1)}, \tag{36}$$

$$T_3^{(1)} = qv - ur,\ X_3^{(1)} = uv_{21}^{(1)} - vv_{12}^{(1)} - 2v^{(2)}. \tag{37}$$

The first two conservation laws are obvious consequence of the divergent form of equations (11).

Let us discuss in the case of system of nonlinear equations

$$iq_t + q_{xx} - i(q^2r)_x = 0, \tag{38}$$

$$ir_t - r_{xx} - i(qr^2)_x = 0, \tag{39}$$

the connection between solutions (29) and (30) of linearized equations and the conservation laws found. Note that DNLSE (15) follows this system by imposing the condition $r = \pm q^*$. The coefficients of the second equation of Lax pair (1, 2) of equations (38, 39) have the form

$$V^{(4)} = 2U^{(2)}, \quad V^{(3)} = 2U^{(1)}, \quad V^{(2)} = qrU^{(2)},$$

$$V^{(1)} = \begin{pmatrix} 0 & iq_x + q^2r \\ -ir_x + qr^2 & 0 \end{pmatrix}.$$

In the terms of potentials $u$ and $v$ the Lagrangian of system (38, 39) reads as

$$\mathcal{L} = i(u_xv_t + v_xu_t) + u_{xx}v_x - v_{xx}u_x - iu_x^2v_x^2\,.$$

Using the designations for the Euler–Lagrange equations

$$\begin{aligned}\Lambda(u) &\equiv -\left(\frac{\partial\mathcal{L}}{\partial u_t}\right)_t - \left(\frac{\partial\mathcal{L}}{\partial u_x}\right)_x + \left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right)_{xx} \\ &= -2(ir_t - r_{xx} - i(qr^2)_x) = 0, \Lambda(v) \\ &\equiv -\left(\frac{\partial\mathcal{L}}{\partial v_t}\right)_t - \left(\frac{\partial\mathcal{L}}{\partial v_x}\right)_x + \left(\frac{\partial\mathcal{L}}{\partial v_{xx}}\right)_{xx} \\ &= -2(iq_t + q_{xx} - i(q^2r)_x) = 0,\end{aligned}$$

the variation of the Lagrangian, which is caused by the infinitesimal transformations of potentials $u \to u + \varepsilon\delta u$ and $v \to v + \varepsilon\delta v$, is written in the form of Noether's identity

$$\delta\mathcal{L} = \varepsilon(A_t + B_x + \Lambda(u)\delta u + \Lambda(v)\delta v). \tag{40}$$

Here

$$\begin{aligned}A &= \frac{\partial\mathcal{L}}{\partial u_t}\,\delta u + \frac{\partial\mathcal{L}}{\partial v_t}\,\delta v = i(q\,\delta v + r\,\delta u), \\ B &= \left(\frac{\partial\mathcal{L}}{\partial u_x} - \left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right)_x\right)\delta u + \frac{\partial\mathcal{L}}{\partial u_{xx}}\,\delta u_x \\ &\quad + \left(\frac{\partial\mathcal{L}}{\partial v_x} - \left(\frac{\partial\mathcal{L}}{\partial v_{xx}}\right)_x\right)\delta v + \frac{\partial\mathcal{L}}{\partial v_{xx}}\,\delta v_x \\ &= q_x\,\delta v - r_x\,\delta u + r\,\delta q - q\,\delta r - iqr^2\,\delta u - iq^2r\,\delta v.\end{aligned}$$

Given a symmetry of equations (38, 39), a conservation law is derived from equation (40) due to the Noether's theorem. Few examples of the symmetries and associated conservation densities and currents are listed below:

**(1)** $\delta u = 1$, $\delta v = 0$:

$$\delta\mathcal{L} = 0,$$

$$T_1 = ir, \quad X_1 = iv_t - 2r_x - 2iqr^2. \tag{41}$$

**(2)** $\delta u = 0$, $\delta v = 1$:

$$\delta\mathcal{L} = 0,$$

$$T_2 = iq, \quad X_2 = iu_t + 2q_x - 2iq^2r. \tag{42}$$

**(3)** $u \to u\mathrm{e}^{i\varepsilon}$, $v \to v\mathrm{e}^{-i\varepsilon}$, $\delta u = iu$, $\delta v = -iv$:

$$\delta\mathcal{L} = 0,$$

$$T_3 = qv - ur,$$
$$X_3 = u_tv - uv_t - 2i(q_xv - qr + ur_x) - 2(qv - ur)qr. \tag{43}$$

**(4)** $x \to x + \varepsilon$, $\delta u = q$, $\delta v = r$:

$$\delta\mathcal{L} = \varepsilon\mathcal{L}_x,$$

$$T_4 = 2iqr, \quad X_4 = 2(q_xr - qr_x) - 3iq^2r^2. \tag{44}$$

**(5)** $t \to t + \varepsilon$, $\delta u = u_t$, $\delta v = v_t$:

$$\delta\mathcal{L} = \varepsilon\mathcal{L}_t,$$

$$T_5 = -q_xr + qr_x + iq^2r^2,$$
$$X_5 = i(q_{xx}r - 2q_xr_x + qr_{xx} - 2q^3r^3) + 3(q_xr - qr_x)qr. \tag{45}$$

The conservation laws that arise in the first and second cases are trivial. The symmetries of the potentials in the third, fourth and fifth cases correspond, respectively, to the infinitesimal symmetries $\delta q_1^{(1)}$, $\delta r_1^{(1)}$, $\delta q^{(1)}$, $\delta r^{(1)}$ and $\delta q^{(2)}$, $\delta r^{(2)}$ presented at the end of previous section. Conserved density $T_5$ is proportional to the Hamiltonian density of DNLSE [22]. The Noether's theorem was applied in [49] to obtain $T_3$ and $X_3$, which are nothing but $T_3^{(1)}$ and $X_3^{(1)}$ (37). Hence, $\delta r_1^{(1)}$ and $\delta q^{(1)}$ are connected by the Noether's theorem with $T_3^{(1)}$ and $X_3^{(1)}$. We shall prove below that this is valid for all members of the hierarchies of infinitesimal symmetries and conservation laws.

Formulas (29, 30) give us solutions of the linearized equations on potentials

$$\delta u = \varphi_1^{(1)}\varphi_1^{(2)}, \quad \delta v = -\varphi_2^{(1)}\varphi_2^{(2)}.$$

It is remarkable that we are able to put the corresponding variation of Lagrangian in divergent form:

$$\delta\mathcal{L} = \Big(iu\,\delta r + iv\,\delta q + 4\mu(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)})\Big)_t$$
$$+ \Big(v\,\delta q_x - u\,\delta r_x + q\,\delta r - r\,\delta q - i(ur+2qv)r\,\delta q$$
$$- i(qv+2ur)q\,\delta r + 8i\mu^2(r\delta u + q\delta v)$$
$$- 16\mu^3(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)})\Big)_x.$$

Combining this expression with equation (40), we come after a cancellation of the terms with potentials $u$ and $v$ to the conservation law, whose conserved density $\tilde{T}$ and current $\tilde{X}$ are defined in the following manner

$$\tilde{T} = 4\mu(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)}) + 2i(q\varphi_2^{(1)}\varphi_2^{(2)} - r\varphi_1^{(1)}\varphi_1^{(2)}),$$
$$\tilde{X} = -(16\mu^3 + 4\mu qr)(\varphi_1^{(1)}\varphi_2^{(2)} + \varphi_2^{(1)}\varphi_1^{(2)})$$
$$+ 12i\mu^2(r\varphi_1^{(1)}\varphi_1^{(2)} - q\varphi_2^{(1)}\varphi_2^{(2)})$$
$$+ 2(r_x + iqr^2)\varphi_1^{(1)}\varphi_1^{(2)} + 2(q_x - iq^2r)\varphi_2^{(1)}\varphi_2^{(2)}).$$

These expressions are proportional to ones given by equations (34, 35). This way, we show that solutions (29, 30) of linearized equations and conserved densities (34) and currents (35) are connected in the case of DNLSE in accordance with the Noether's theorem. This connection takes also place between the infinite hierarchies of infinitesimal symmetries $\delta q^{(k)}$, $\delta r^{(k)}$ and $\delta q_j^{(k)}$, $\delta r_j^{(k)}$ ($j = 1, 2, 3$, $k = 0, 1, ...$) and the hierarchies of conservation laws obtained by expansion in formulas (34, 35) of the Lax pair solutions on the spectral parameter powers. First terms of expansions (16) and (17) lead to the conservation laws determined by formulas (44, 45) and (36, 37), respectively, that coincide with ones presented in [22,29,34].

## 5 Conclusion

In the present report, we have found the expressions for the solution of linearization of the DNLSE hierarchy equations and their conservation law in the terms of the solutions of associated Lax pairs. The approach exploited is based on the Darboux transformation technique. It is shown in the DNLSE case that the conservation law is connected with the solution of linearized equation accordingly to the Noether's theorem. The local hierarchy and three nonlocal ones of the infinitesimal symmetries and conservation laws that are explicitly expressed through the variables of the nonlinear equations are produced using the recursion operator and expanding the Lax pair solutions in the series on the spectral parameter powers.

The explicit form of infinitesimal symmetries and conservation laws of various hierarchies is useful to determine an integrability of the nonlinear PDE's given. This is especially important in the cases interesting from the physical point of view, such as the reductions of PDE's and their deformations. Recently, it was revealed that some deformations of the well-known nonlinear integrable equations, which have the physical meaning, are also integrable [15,41]. This opens the problems of a description of classes of the deformations keeping the integrability and an extension to them of the methods having been developed in the IST theory. The approach suggested here is not specific for the hierarchy considered and can be applied to other integrable hierarchies and their integrable deformations. An investigation of the hierarchy of the deformed nonlinear equations, which is associated with the quadratic bundle and contains as a particular case the following integrable deformation of the DNLSE

$$iq_t + \alpha q_x^* + q_{xx} \pm i(|q|^2q)_x = 0,$$

where $\alpha$ is an arbitrary parameter, is a subject of the future work.

It will be also interesting to extend this approach on the multidimensional and nonlocal integrable equations. An existence of the closed differential 1-forms defined on the solutions of corresponding Lax pairs will play important role in these cases. In particular, the DT technique gives the expressions for the solutions of linearized equations, which contain these forms [46,48]. As in (1 + 1)-dimensional case considered here, the solutions of the Lax pairs can also be expanded in the series (at least, for some classes of the initial value problems).

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